Actions of the Dipper-Donkin quantization GL_2 on the Clifford algebra $C(1,3)$.

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Abstract. Following the method already developed for studying the actions of $GL_q(2,C)$ on the Clifford algebra $C(1,3)$ and its quantum invariants [1], we study the action on $C(1,3)$ of the quantum GL_2 constructed by Dipper and Donkin [2]. We are able of proving that there exits only two non-equivalent cases of actions with nontrivial "perturbation" [1]. The spaces of invariants are trivial in both cases.

We also prove that each irreducible finite dimensional algebra representation of the quantum GL_2, $q^m \neq 1$, is one dimensional.

By studying the cases with zero "perturbation" we find that the cases with nonzero "perturbation" are the only ones with maximal possible dimension for the operator algebra $\Re$.

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1 Introduction.

In this paper we consider inner actions of the Dipper-Donkin quantization of GL_2 (see [2]) on the space-time Clifford algebra $C(1,3)$. The analogous problems for Manin quantization was considered in details in [1]. We prove that every irreducible finite dimensional algebra representation of GL_2, $q \neq 1$, is one dimensional and therefore triangular. Using this fact we show that only two particular cases have nonzero *perturbation*. Actions with trivial *perturbation* are also studied. From this, some consequences are derived.

This paper is organized as follows. In Section 2, we introduce elementary notions. In Section 3 we prove Theorem 1 which is fundamental to address our problem. This Theorem deals with q-spinor representations ($q^3 \neq 1$, $q^4 \neq 1$), corresponding to representations of GL_2 with non zero *perturbation*. Since each irreducible finite dimensional algebra representation of Dipper-Donkin GL_2 is one dimensional, we can use the method of [1] in complete generality for the classification of inner actions. Finally, in Section 4 we present the representations of GL_2 by Dipper-Donkin with nonzero *perturbation* and some remarkable features related with the zero *perturbation* cases.

2 Preliminary notions.

The algebraic structure of Dipper-Donkin quantization GL_2 [2] is generated by four elements c_{ij}, $1 \leq i, j \leq 2$ with relations which can be presented by the following diagram.

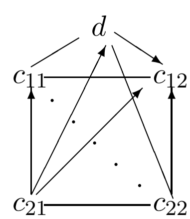

Figure 1. GL_2

Here we denote by arrows the "quantum spinors" (or generators of the quantum plane [3]) $xy = qyx$ by straigth line the "classical spinors" $xy = yx$ [1] and by dots a classical spinor with a nontrivial *perturbation* [1], $xy - yx = p$ being $p{=}(q-1)c_{12}c_{21}$.

Here the quantum determinant $d = c_{11}c_{22} - c_{12}c_{21}$ is noncentral and group-like. This, in contrast with Manin's approach [3]. In any Hopf algebra every group-like element is invertible, therefore the quantum GL_2 includes the formal inverse d^{-1}.

The coalgebra structure is defined in the standard way for all quantizations and the antipode S is given in reference [2].

The Clifford algebra $C(1,3)$ is generated by the vector γ_μ, $\mu = 0, 1, 2, 3$ with relations defined by the form $g_{\mu\nu}{=}diag(1, -1, -1, -1)$, as follows:

$$\gamma_\mu \gamma_\nu = g_{\mu\nu} + \gamma_{\mu\nu}, \quad \gamma_{\mu\nu} = -\gamma_{\nu\mu},$$

$$\gamma_\rho \gamma_{\mu\nu} = g_{\rho\mu}\gamma_\nu - g_{\rho\nu}\gamma_\mu + \gamma_{\rho\mu\nu},$$

$$\gamma_\lambda \gamma_{\mu\nu\rho} = g_{\lambda\mu}\gamma_{\nu\rho} - g_{\lambda\nu}\gamma_{\mu\rho} + g_{\lambda\rho}\gamma_{\mu\nu} + \gamma_{\lambda\mu\nu\rho}.$$

This algebra is isomorphic to the algebra of the 4×4 complex matrix and it has the basis of matrix units reported in reference [1], among others.

An action of GL_2 on $C(1,3)$ is uniquely defined by actions of c_{ij} on the generators of $C(1,3)$[4][5];

$$c_{ij} \cdot \gamma_k = f_{ijk}(\gamma_0, \gamma_1, \gamma_2, \gamma_3), \tag{1}$$

where f_{ijk} are some noncommutative polynomials in four variables. If (1) defines an action of quantum group GL_2 on $C(1,3)$ and γ_0', γ_1', γ_2', γ_3' is another system of generators of $C(1,3)$, with the same relations, then the formula

$$c_{ij} * \gamma_k' = f_{ijk}(\gamma_0', \gamma_1', \gamma_2', \gamma_3'), \tag{2}$$

with the same polynomials f_{ijk}, will also define an action of the quantum GL_2 on $C(1,3)$. Two actions of GL_2 on $C(1,3)$ are said to be equivalent if they can be presented as in (1) and (2) with the same polynomials f_{ijk}. It is easy to show that two actions $\cdot$, $*$ are equivalent if and only if $c_{ij} * (uwu^{-1}){=}\ u(c_{ij} \cdot w)u^{-1}$ for some invertible $u \in C(1,3)$ (see [1], formula (7)).

For every action $\cdot$ there exist an invertible matrix $M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$ $\in C(1,3)_{2\times 2}$, such that

$$c_{ij} \cdot v = \sum m_{ik} v m^*_{kj},$$

where $\begin{pmatrix} m^*_{11} & m^*_{12} \\ m^*_{11} & m^*_{12} \end{pmatrix}$$=M^{-1}$ (see Skolem-Noether theorem for Hopf algebras [6][7]). The action $\cdot$ is called inner if the map $c_{ij} \to m_{ij}$ defines an algebra homorphism $\varphi : GL_2 \to C(1,3)$. Since the algebra $C(1,3)$ is isomorphic to the algebra of 4×4 matrices, the homorphism $C(1,3)$ defines (and is defined by) a four dimensional module over (the algebraic structure of) GL_2, or, equivalently, a four dimensional representation of GL_2.

If $\varphi(c_{12}c_{21})$=0, then by definition in Figure 1 the representation φ is defined for an essentially more simple structure, generated by two commuting "quantum spinors" (c_{21}, c_{11}) and (c_{22}, c_{12}). Firstable we focus our attention on the case when $\varphi(c_{12}c_{21})\neq 0$ and in this case we say that the inner action defined by φ has nonzero *perturbation*.

If we add a formal inverse c_{11}^{-1}, then the algebraic structure of Dipper-Donkin quantization GL_2 is generated by the elements in the following diagram.

$$\begin{tikzcd}
c_{11} \arrow[r, dash] \arrow[dr, dash] & a_{12} = c_{11}^{-1} c_{12} \\
c_{21} \arrow[u] \arrow[r, dash] \arrow[ur, dash] & a_{22} = c_{11}^{-1} d \arrow[u]
\end{tikzcd}$$

Figure 2. GL_2

From here, it follows straightforward that, up to invertibility of c_{11}, the algebraic structure of GL_2 can be considered like a tensor product $\aleph \otimes \aleph$ where $\aleph$ is the quantum plane.

In the next Section we study q-spinors suitable of being used to represent

the quantum GL_2. Concretely speaking we consider in details the following triangle.

$$\begin{tikzcd} A \arrow[r] & B \\ C \arrow[u] \arrow[ur] & \end{tikzcd}$$

Figure 3.

corresponding to

$$\begin{tikzcd} d \arrow[r] & c_{12} \\ c_{21} \arrow[u] \arrow[ur] & \end{tikzcd}$$

Figure 4.

in Figure 1.

We say that the representation of the q-spinor $xy{=}qyx$, $x \to A$, $y \to B$ is *admissible* if there exists C such that $x \to C$, $y \to B$ and $x \to C$, $y \to A$ are also a representation of q-spinor with $CB \neq 0$. In other words it means that $d \to A$, $c_{12} \to B$, $c_{21} \to C$ is a representation of the subalgebra of GL_2, generated by d, c_{12}, c_{21} with $CB \neq 0$.

3 q-spinor representations.

Let (x, y) be a q-spinor, $xy = qyx$. If $x \to A$, $y \to B$ is its representation by 4×4 matrices over complex numbers, then for every invertible 4×4 matrix u and nonzero number α, the map $x \to uAu^{-1}\alpha$, $y \to uBu^{-1}\alpha$ also defines a representation of the q-spinor. Following, [1], we consider

this two representations as *equivalent* ones. Thus, under investigation of representations of a q-spinor, we can suppose that the matrix A has a Jordan Normal form and one of it's eigenvalues is equal to 1 (if $A \neq 0$).

For a given matrix A, we denote by $B(A)$ the linear space of all matrices B, such that $AB{=}qBA$ and by $B'(A)$ the set of all matrices B' such that $B'A{=}qAB'$.

Theorem 1. *Every admissible representation of the q-spinor* $(q^3, q^4 \neq 1)$ *[8][9] by* 4×4 *complex matrices,* $x \to A$*,* $y \to B$*, such that* A *is an invertible matrix is equivalent to one of the following representations.*

$$1. \quad A = diag(q^2, q, q, 1), \qquad B = qe_{13} - \mu e_{24} \tag{3}$$

$$B' = e_{43} - \mu e_{21}$$

$$2. \quad A = diag(q^2, q, q, 1), \qquad B = qe_{12} + \mu e_{34} \tag{4}$$

$$B' = e_{42} + \mu e_{31}$$

$$3. \quad A = diag\left(\left(\begin{array}{cc} q & 1 \\ 0 & q \end{array}\right), q^2, 1\right), \qquad B_1 = e_{14} \; ; \; B_2 = e_{32} \tag{5}$$

$$B_1' = e_{13} \; ; \; B_2' = e_{42}$$

Proof. If $B(A)^2 \neq 0$; then by Theorem 1 [1], we have seven different possibilities for A. Direct calculations show that only in the second case there exist representations with nonzero perturbation, these are (3) and (4) described in the theorem.

Let us now study the case $B(A)^2{=}0$. We assume that the matrix A has a Jordan Normal form and one of its eigenvalues is equal to 1. By lemma 2 [1] the matrix A cannot be a simplest Jordan Normal matrix; i.e. it has more than one block.

If $A{=}diag(\alpha_1, \alpha_2, \alpha_3, 1)$ is a diagonal matrix then $B'(A)$ evidentely coincides with the space of transposed matrices $B(A)^T$. By Lemma 4 [1] the space $B(A)$ as well as $B(A)'$ are generated by matrix units and, by formula (20) [1], $e_{ij} \in B(A)$ if and only if $\alpha_i{=}q\alpha_j$, (21) in [1]. Thus we have two main

cases with $B(A)^2$=0; α_1=$q\alpha_2$, α_3=q; e_{12}, $e_{34}\in$ $B(A)$ and α_1=α_2=α_3=q; e_{12}, e_{13}, $e_{14}\in$ $B(A)$ while the others can be obtained from these by changing the numerations of indeces. In both cases $\Big(B(A)\cdot B(A)^T\Big)\cap\Big(B(A)^T\cdot B(A)\Big)$=0 and so there is no admissible representations.

Let A be of the form

$$A=\begin{pmatrix} a & 0\\ 0 & b\end{pmatrix} \tag{6}$$

where a, b are either invertible 2×2 matrices in Jordan Normal form or a is an invertible simplest Normal Jordan 3×3 matrix and b is a nonzero complex number (and therefore we can suppose that b=1).

If B'= $\begin{pmatrix} \alpha' & \beta'\\ \gamma' & \delta'\end{pmatrix}$ is a nonzero matrix from $B'(A)$ then by formula (23) [1] changing q by q^{-1} we have that

$$a\alpha' = q^{-1}\alpha' a \qquad a\beta' = q^{-1}\beta' b \tag{7}$$

$$b\gamma' = q^{-1}\gamma' a \qquad b\delta' = q^{-1}\delta' b. \tag{8}$$

At first, let us consider when a is a 3×3 matrix. In [1] we can see that in this case there exists only two possibilities with $B(A)\neq 0$;

$$A=\begin{pmatrix} q^{-1} & 1 & 0 & 0\\ 0 & q^{-1} & 1 & 0\\ 0 & 0 & q^{-1} & 0\\ 0 & 0 & 0 & 1\end{pmatrix},\quad B(A)=Ce_{43}$$

and

$$A=\begin{pmatrix} q & 1 & 0 & 0\\ 0 & q & 1 & 0\\ 0 & 0 & q & 0\\ 0 & 0 & 0 & 1\end{pmatrix},\quad B(A)=Ce_{14}.$$

In the first case, we have $B'(A)$=Ce_{14} and in the second $B'(A)$=Ce_{43}. Thus the equality $Ce_{43}\cdot Ce_{14}$=0 shows that in both cases either $B(A)B'(A)$=0 or $B'(A)B(A)=0$ and there exits no admissible representation.

Consider now the case when $a, b, \alpha, \beta, \gamma, \delta$ are 2×2 matrices. Here, we have defined $B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$.

Let us start with the case when both matrices a and b have a simplest Jordan Normal Form i.e.

$$a = \begin{pmatrix} \epsilon & 1 \\ 0 & \epsilon \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{9}$$

(recall that we suppose that one of the eigenvalues of A=diag(a, b) is equal to 1).

We know that $[A, B]_q$=0 (from this follows that α= $\delta = 0$) and $[A, B']_{q^{-1}} = 0$ (from this follows that α'= $\delta' = 0$). See Lemma 2 in reference [1]. Besides we require

$$BB' = \begin{pmatrix} \beta\gamma' & 0 \\ 0 & \gamma\beta' \end{pmatrix} = q \begin{pmatrix} \beta'\gamma & 0 \\ 0 & \gamma'\beta \end{pmatrix} = qBB'. \tag{10}$$

Therefore the following formulas must be fulfilled.

$$b\gamma' = q^{-1}\gamma' a \qquad a\beta' = q^{-1}\beta' b \tag{11}$$

$$a\beta = q\beta b \qquad b\gamma = q\gamma a \tag{12}$$

$$\beta\gamma' = q\beta'\gamma \qquad \gamma\beta' = q\gamma'\beta. \tag{13}$$

We have two cases.

I) For ϵ=q

$$\beta = \begin{pmatrix} \beta_{11} & \beta_{12} \\ 0 & q\beta_{11} \end{pmatrix}; \gamma = \beta' = 0; \gamma' = \begin{pmatrix} \gamma'_{11} & \gamma'_{12} \\ 0 & q^{-1}\gamma'_{11} \end{pmatrix}. \tag{14}$$

Formulas (11) and (12) follow straightforward, to fulfill (13) we require

$$a)\ \beta = 0, \quad \gamma' = \begin{pmatrix} \gamma'_{11} & \gamma'_{12} \\ 0 & q^{-1}\gamma'_{11} \end{pmatrix} \quad \text{or} \tag{15}$$

$$b)\ \beta = \begin{pmatrix} \beta_{11} & \beta_{12} \\ 0 & q\beta_{11} \end{pmatrix}, \quad \gamma' = 0\,. \tag{16}$$

In this case we conclude that either

$$a)\ \beta = \begin{pmatrix} \beta_{11} & \beta_{12} \\ 0 & q\beta_{11} \end{pmatrix}, \quad \gamma = 0, \quad \beta' = 0, \text{ and } \gamma' = 0\ , \tag{17}$$

this means $B' = 0$ and $BB' = 0$, or

$$b)\ \beta = 0, \quad \gamma' = \begin{pmatrix} \gamma'_{11} & \gamma'_{12} \\ 0 & q^{-1}\gamma'_{11} \end{pmatrix}, \quad \gamma = 0 \text{ and } \beta' = 0. \tag{18}$$

This also means $BB' = 0$. II) For ϵ=q^{-1}

$$\beta' = \begin{pmatrix} \beta'_{11} & \beta'_{12} \\ 0 & q^{-1}\beta'_{11} \end{pmatrix}; \gamma' = \beta = 0; \gamma = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ 0 & q\gamma_{11} \end{pmatrix}\ . \tag{19}$$

Formulas (11) and (12) follow straightforward, to fulfill (13) we require

$$a)\ \beta' = 0, \quad \gamma = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ 0 & q\gamma_{11} \end{pmatrix} \tag{20}$$

$$b)\ \beta' = \begin{pmatrix} \beta'_{11} & \beta'_{12} \\ 0 & q^{-1}\beta'_{11} \end{pmatrix}, \ \gamma = 0\ . \tag{21}$$

In this case we conclude that either

$$a)\ \beta' = \begin{pmatrix} \beta'_{11} & \beta'_{12} \\ 0 & q^{-1}\beta'_{11} \end{pmatrix}, \quad \gamma = 0, \quad \beta = 0, \text{ and } \gamma' = 0\ , \tag{22}$$

this means $B = 0$ and $BB' = 0$, or

$$b)\ \beta = 0, \quad \gamma = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ 0 & q\gamma_{11} \end{pmatrix}, \quad \gamma' = 0 \text{ and } \beta' = 0. \tag{23}$$

This means $B' = 0$ and $BB' = 0$.

Suppose now that one of the matrix a, b is a simplest Jordan matrix while the other is a diagonal matrix. A conjugation by $T = \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix}$, where E is the identity 2×2 matrix, changes A=diag(a, b) to diag(b, a), so we can suppose that

$$a = \epsilon E + e_{12}, \quad b = diag(\mu, 1) \tag{24}$$

(recall that one of the eigenvalues of A is equal to 1 and A is an invertible matrix; i.e. $\epsilon, \mu \neq 0$).

Firstable, let $\mu \neq q, q^{-1}$, $\delta' = 0$, then

$$B = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}; \quad B' = \begin{pmatrix} 0 & \beta' \\ \gamma' & 0 \end{pmatrix} \tag{25}$$

We set $BB' = qB'B$ and require formulas (11)- (12) to hold. From this we obtain the following four cases.

I) For β;

$$\begin{aligned}
&A) \quad \beta = 0 \quad \text{provided } \epsilon \neq q\mu, \quad \epsilon \neq q, \\
&B) \quad \beta = \begin{pmatrix} 0 & \beta_{12} \\ 0 & 0 \end{pmatrix} \text{ provided } \epsilon \neq q\mu, \quad \epsilon = q, \\
&C) \quad \beta = \begin{pmatrix} \beta_{11} & 0 \\ 0 & 0 \end{pmatrix} \text{ provided } \epsilon = q\mu, \quad \epsilon \neq q, \\
&D) \quad \beta = \begin{pmatrix} \beta_{11} & \beta_{12} \\ 0 & 0 \end{pmatrix} \text{ provided } \epsilon = q\mu, \quad \epsilon = q.
\end{aligned}$$

II) For γ,

$$\begin{aligned}
&A) \quad \gamma = 0 \quad \text{provided } \epsilon \neq q^{-1}\mu, \quad \epsilon \neq q^{-1}, \\
&B) \quad \gamma = \begin{pmatrix} 0 & 0 \\ 0 & \gamma_{22} \end{pmatrix} \text{ provided } \epsilon \neq q^{-1}\mu, \quad \epsilon = q^{-1}, \\
&C) \quad \gamma = \begin{pmatrix} 0 & \gamma_{12} \\ 0 & 0 \end{pmatrix} \text{ provided } \epsilon = q^{-1}\mu, \quad \epsilon \neq q^{-1}, \\
&D) \quad \gamma = \begin{pmatrix} 0 & \gamma_{12} \\ 0 & \gamma_{22} \end{pmatrix} \text{ provided } \epsilon = q^{-1}\mu, \quad \epsilon = q^{-1}.
\end{aligned}$$

III) For β',

$$\begin{aligned}
&A) \quad \beta' = 0 \quad \text{provided } \epsilon \neq q^{-1}\mu, \quad \epsilon \neq q^{-1}, \\
&B) \quad \beta' = \begin{pmatrix} 0 & \beta'_{12} \\ 0 & 0 \end{pmatrix} \text{ provided } \epsilon \neq q^{-1}\mu, \quad \epsilon = q, \\
&C) \quad \beta = \begin{pmatrix} \beta'_{11} & 0 \\ 0 & 0 \end{pmatrix} \text{ provided } \epsilon = q^{-1}\mu, \quad \epsilon \neq q^{-1},
\end{aligned}$$

$$D) \quad \beta' = \begin{pmatrix} \beta'_{11} & \beta'_{12} \\ 0 & 0 \end{pmatrix} \text{ provided } \epsilon = q^{-1}\mu, \quad \epsilon = q^{-1}.$$

IV) For γ',

$$\begin{aligned}
&A) \quad \gamma' = 0 \quad \text{provided } \epsilon \neq q\mu, \quad \epsilon \neq q, \\
&B) \quad \gamma = \begin{pmatrix} 0 & 0 \\ 0 & \gamma'_{22} \end{pmatrix} \text{ provided } \epsilon \neq q\mu, \quad \epsilon = q, \\
&C) \quad \gamma = \begin{pmatrix} 0 & \gamma'_{12} \\ 0 & 0 \end{pmatrix} \text{ provided } \epsilon = q\mu, \quad \epsilon \neq q. \\
&D) \quad \gamma \begin{pmatrix} 0 & \gamma'_{12} \\ 0 & \gamma'_{22} \end{pmatrix} \text{ provided } \epsilon = q\mu, \quad \epsilon = q
\end{aligned}$$

We can reorganize cases I-IV in the following way.

i) Let $\epsilon \neq q\mu$, $\epsilon \neq q$. Then we have

$$\begin{aligned}
&i.1) \quad \beta'_{12} = 0 \;\text{ or }\; \gamma_{22} = 0 \;\text{ for }\; \epsilon \neq q^{-1}\mu; \epsilon = q^{-1}; \epsilon \neq q\mu, \\
&i.2) \quad \beta'_{11} = 0 \;\text{ or }\; \gamma_{12} = 0 \;\text{ for }\; \epsilon = q^{-1}\mu; \epsilon \neq q^{-1}; \epsilon \neq q\mu, \\
&i.3) \quad \gamma_{12}\beta'_{11} = -\beta'_{12}\gamma_{22} \;\text{ for }\; \epsilon = q^{-1}\mu; \epsilon = q^{-1}, \\
&i.4) \quad \text{No extra condition for } \;\epsilon \neq q^{-1}\mu;\; \epsilon \neq q^{-1};\; \epsilon \neq q\mu; \epsilon \neq q.
\end{aligned}$$

Studying $i.1)$ we obtain two possible cases.

$$a) \; \beta' = 0 \;, \; B' = 0$$

$$b) \; \gamma = 0 \;, \; B' = 0.$$

In both cases BB'=0.

For $i.2)$ we again obtain two possible cases

$$a) \; \beta'_{11} = 0 \quad \text{then } B' = 0.$$

$$b) \; \gamma_{12} = 0 \quad \text{then } B = 0.$$

In both cases BB'=0.

For $i.3)$ we obtain,

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \gamma_{12} & 0 & 0 \\ 0 & \gamma_{22} & 0 & 0 \end{pmatrix} \text{ and } B' = \begin{pmatrix} 0 & 0 & \beta'_{11} & \beta'_{12} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

From this, we conclude again that BB'=$B'B$=0.

In the case $i.4)$ we have β=γ=β'= γ'=0, thus BB'=0.

ii) Let $\epsilon \neq q\mu$, $\epsilon = q$. In this case we have

$$B = \begin{pmatrix} 0 & 0 & 0 & \beta_{12} \\ 0 & 0 & 0 & 0 \\ 0 & \gamma_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } B' = \begin{pmatrix} 0 & 0 & \beta'_{11} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \gamma'_{22} & 0 & 0 \end{pmatrix}$$

From this follows that

$$A = diag\left(\begin{pmatrix} q & 1 \\ 0 & q \end{pmatrix}, \begin{pmatrix} q^2 & 0 \\ 0 & 1 \end{pmatrix}\right) \quad B_1 = e_{14},\ B_2 = e_{32},\ B'_1 = e_{13}, B'_2 = e_{42}$$

which corresponds to representation (5) in Theorem 1.

iii) Let $\epsilon \neq q\mu$, $\epsilon \neq q$. In this case we have

$$B = \begin{pmatrix} 0 & 0 & \beta_{11} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \gamma_{22} & 0 & 0 \end{pmatrix} \text{ and } B' = \begin{pmatrix} 0 & 0 & 0 & \beta'_{12} \\ 0 & 0 & 0 & 0 \\ 0 & \gamma'_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

From this follows that

$$A = diag\left(\begin{pmatrix} q^{-1} & 1 \\ 0 & q^{-1} \end{pmatrix}, \begin{pmatrix} q^{-2} & 0 \\ 0 & 1 \end{pmatrix}\right),\ B_1 = e_{13},\ B_2 = e_{42},\ B'_1 = e_{14}, B'_2 = e_{32}$$

By applying the maps $q \rightarrow q^{-1}$ and $B \rightarrow B'$ we obtain the representation (5) in Theorem 1.

iv). Let $\mu = 1$ (namely ϵ=$q\mu$, ϵ=q). In this case we have

$$B = \begin{pmatrix} 0 & 0 & \beta_{11} & \beta_{12} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } B' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \gamma'_{12} & 0 & 0 \\ 0 & \gamma'_{22} & 0 & 0 \end{pmatrix}.$$

From (13), follows that BB'=0, since $\beta_{11}\gamma'_{12}+\beta_{12}\gamma'_{22}=0$.

Let us now consider the case $\mu=q^{-1}$, we can multiply matrices A and B by q and conjugate them by the matrix

$$diag(1,1,\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}).$$

We will obtain an equivalent representation with $\mu=q$. Thus, it is enough to consider the case

$$a=\epsilon E+e_{12}, \quad b=diag(q,1)$$

where

$$\alpha=0, \quad \delta=ce_{12}, \quad c\in\mathbf{C}.$$

For $\beta=\begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix}$, then we have $\alpha\beta=q\beta b$ e; i.e.

$$\begin{pmatrix} \epsilon & 1 \\ 0 & \epsilon \end{pmatrix}\begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix}=\begin{pmatrix} \epsilon\beta_{11}+\beta_{21} & \epsilon\beta_{12}\beta_{22} \\ \epsilon\beta_{21} & \epsilon\beta_{22} \end{pmatrix}=$$

$$q\begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix}\begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}=q\begin{pmatrix} q\beta_{11} & \beta_{12} \\ q\beta_{21} & \beta_{22} \end{pmatrix},$$

which implies

$$(\epsilon-q^2)\beta_{11}=-\beta_{21},(\epsilon-q)\beta_{12}=-\beta_{22} \tag{26}$$

$$(\epsilon-q^2)\beta_{21}=0,(\epsilon-q)\beta_{22}=0. \tag{27}$$

If ϵ=q^2 then the first equality of (26) gives $\beta_{21}=0$, and if $\epsilon\neq q^2$ then the first equality of (27) gives $\beta_{21}=0$. Therefore $\beta_{21}=0$ in any case. In the same way $\beta_{22}=0$ and (26), (27) are equivalent to

$$(\epsilon-q^2)\beta_{11}=0,\ (\epsilon-q)\beta_{12}=0, \tag{28}$$

$$\beta_{21}=0,\ \beta_{22}=0. \tag{29}$$

Analogously for the matrix $\gamma = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix}$ we have $b\gamma{=}q\gamma a$; i.e.

$$\begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix} = \begin{pmatrix} q\gamma_{11} & q\gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix} =$$

$$q\begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix}\begin{pmatrix} \epsilon & 1 \\ 0 & \epsilon \end{pmatrix} = q\begin{pmatrix} \epsilon\gamma_{11} & \gamma_{11}+\epsilon\gamma_{12} \\ \epsilon\gamma_{21} & \gamma_{21}+\epsilon\gamma_{22} \end{pmatrix}.$$

This implies

$$q(1-\epsilon)\gamma_{11} = 0,\ q(1-\epsilon)\gamma_{12} = q\gamma_{11} \tag{30}$$

$$(1-q\epsilon)\gamma_{21} = 0,\ (1-q\epsilon)\gamma_{22} = q\gamma_{21}. \tag{31}$$

Again, if $\epsilon = 1$ then by the second equality of (30), $\gamma_{11} = 0$ and if $\epsilon \neq 1$ then by the first one $\gamma_{11} = 0$. In the same way $\gamma_{21} = 0$ and (30),(31) are equivalent to

$$\gamma_{11} = 0, \quad (1-\epsilon)\gamma_{12} = 0, \tag{32}$$

$$\gamma_{21} = 0, \quad (1-q\epsilon)\gamma_{22} = 0. \tag{33}$$

Now if $\epsilon \neq q^{-1}, 1, q, q^2$ then by (28), (29) and (32), (33) $\beta = \gamma = 0$ and the representation has the form

$$A = diag\left(\begin{pmatrix} \epsilon & 1 \\ 0 & \epsilon \end{pmatrix}, q, 1\right), \quad B = e_{34}. \tag{34}$$

In this case $(B(A) \cdot B'(A)) \cap (B(A)' \cdot B(A)) = 0$; namely the representation is not admissible.

Finally, let us consider four last possibilities.

1. $\epsilon{=}q^{-1}$. By (28) and (29) we have $\beta = 0$ and by (32) and (33), $\gamma = ce_{22}$. From this follows that

$$A = diag\left(\begin{pmatrix} q^{-1} & 1 \\ 0 & q^{-1} \end{pmatrix}, q, 1\right), \quad B_1 = e_{42}, \quad B_2 = e_{34}.$$

If we multiply A by q and conjugate it by T= diag$(1, q^{-1}, 1, 1)$ we will obtain an equivalent representation $A{=}diag(1, 1, q^2, q){+}e_{12}$, $B_1{=}\ e_{42}$, $B_2{=}e_{34}$. Using

conjugations by matrices $E - e_{ii} - e_{jj} + e_{ij} + e_{ji}$ we can change indices with the help of permutation $1 \to 3$, $2 \to 4$, $3 \to 1$, $4 \to 2$. Therefore, $e_{42} \to e_{24}$, $e_{34} \to e_{12}$ and we have the representation $A{=}diag(q^2, q, 1, 1){+}e_{34}$, $B_1 = e_{24}$, $B_2 = e_{12}$, $B_1' = e_{21}$, $B_2' = e_{32}$. In this case $(B(A) \cdot B'(A)) \cap (B(A)' \cdot B(A)) = 0$; namely the representation is not admissible.

2. $\epsilon = 1$. By (28) and (29) we again have $\beta = 0$ and by (32) and (33), $\gamma = Ce_{12}$. From this the representation has the following form.

$$A = diag\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, q, 1\right), \quad B = e_{32}, \quad B_2 = e_{34}.$$

and therefore $B(A)^2 = 0$.

3. $\epsilon = q$. By (32) and (33) we have $\gamma = 0$ and by (28) and (29) $\beta = ce_{12}$. From this the representation has the form

$$A = diag\left(\begin{pmatrix} q & 1 \\ 0 & q \end{pmatrix}, q, 1\right), \quad B_1 = e_{14}, \quad B_2 = e_{34}$$

and again $B(A)^2 = 0$.

4. $\epsilon = q^2$. By (32) and (33) we have $\gamma = 0$ and equalities (28) and (29) imply $\beta = ce_{11}$. So the representation has the form

$$A = diag\left(\begin{pmatrix} q^2 & 1 \\ 0 & q^2 \end{pmatrix}, q, 1\right), \quad B_1 = e_{13}, \quad B_2 = e_{34}.$$

In this case $(B(A) \cdot B'(A)) \cap (B(A)' \cdot B(A)) = 0$; namely the representation is not admissible. $\square$.

4 GL_2 representations.

Theorem 2. *Each irreducible finite dimensional algebra representation of the quantum GL_2, $q^m \neq 1$, is one dimensional.*

Proof. Let $c_{ij} \to C_{ij}$ be a finite dimensional irreducible representation of the quantum GL_2, where C_{ij} are $n \times n$ matrices acting on the n-dimensional

space V. This means that the matrices C_{ij} satisfy the relations of GL_2:

$$C_{11}C_{12} = C_{12}C_{11}, \quad C_{21}C_{11} = qC_{11}C_{21}, \tag{35}$$

$$C_{22}C_{12} = qC_{12}C_{22}, \quad C_{21}C_{22} = C_{22}C_{21}, \tag{36}$$

$$C_{21}C_{12} = qC_{12}C_{21}, \quad C_{22}C_{11} - C_{11}C_{22} = (q-1)C_{12}C_{21} \tag{37}$$

and the matrix $det_q = C_{11}C_{22} - C_{12}C_{21}$ is invertible.

From the relations (35)-(37) follow that $C_{12}V$ is an invariant subspace:

$$C_{11}(C_{12}V) = C_{12}(C_{11}V) \subseteq C_{12}V, \tag{38}$$

$$C_{22}(C_{12}V) = qC_{12}C_{22}V = C_{12}(qC_{22}V) \subseteq C_{12}V, \tag{39}$$

$$C_{21}(C_{12}V) = qC_{12}(C_{21}V) = C_{12}(qC_{21}V) \subseteq C_{12}V. \tag{40}$$

Therefore either $C_{12} = 0$ or C_{12} is an invertible matrix. In the same way either $C_{21} = 0$ or C_{21} is invertible.

If both matrices C_{12}, C_{21} are equal to zero, then the matrices C_{11}, C_{22} commute therefore they have a common eigenvector v and Cv is an invariant subspace, so $Cv = V$, $dimV = 1$.

Suppose that $C_{21} = 0$ and C_{12} is invertible. Then $det_q = C_{11}C_{22}$ and both matrices C_{11}, C_{22} are invertible. Now $x \to C_{22}$, $y \to C_{12}$ is a representation of the q-spinor with invertible matrices which is a contradiction. Recall that if $q^m \neq 1$, and one of the matrices in the q-spinor is invertible; then the second one must be nilpotent.

Suppose that $C_{12} = 0$ and C_{21} is invertible. Then $det_q = C_{11}C_{22}$ and both matrices C_{11}, C_{22} are invertible. Now $x \to C_{11}$, $y \to C_{21}$ is a representation of the q^{-1}- spinor with invertible matrices which is a contradiction.

Finally, let C_{12}, C_{21} be invertible matrices and C_{11}, C_{22} be nilpotent ones. We have the following relation

$$[C_{11}, C_{22}] = (q-1)C_{12}C_{21} = \epsilon, \tag{41}$$

here ϵ is an invertible matrix, such that

$$\epsilon C_{11} = qC_{11}\epsilon. \tag{42}$$

Using this relation and induction by k we can prove that,

$$[C_{11}^k, C_{22}] = q^{[k]} C_{11}^{k-1} \epsilon, \tag{43}$$

where

$$x^{[k]} = 1 + x + ... + x^{k-1} = \frac{x^k - 1}{x - 1} = x^{[k-1]} \cdot x + 1. \tag{44}$$

Indeed, if k is the smallest number such that $C_{11}^k = 0$, then (43) gives a contradiction: $[C_{11}^k, C_{22}]$= $q^{[k]} C_{11}^{k-1} \epsilon = 0$ and $C_{11}^{k-1} = 0$ since $q^{[k]}$=$\frac{1-q^k}{1-q^{-2}} \neq 0$ and ϵ is invertible. $\square$.

From this follows straightforward that every finite dimensional representation of the quantum $GL_2, q^m \neq 1$, is triangular; i.e. it is equivalent to a representation by triangular matrices $c_{ij} \to C_{ij}$.

Corollary 1. *For every finite dimensional representation $c_{ij} \to C_{ij}$ of the quantum $GL_2, q^m \neq 1$, the elements C_{11}, C_{22} are invertible, while C_{12}, C_{21} are nilpotent.*

Proof. We can suppose that C_{ij} are triangular matrices. In this case the matrix

$$(1 - q)^{-1}(C_{11}C_{22} - C_{22}C_{11}) \tag{45}$$

has only zero entries on the main diagonal. This matrix is equal to $C_{12}C_{21}$. From this follows that the main diagonal of $C_{11}C_{22}$ and that of the invertible matrix det_q=$C_{11}C_{22} - C_{12}C_{21}$ coincide. This means that C_{11} and C_{22} have no zero terms on the main diagonal and therefore they are invertible. $\square$

Theorem 3. *Let $c_{ij} \to C_{ij}$ and $c_{ij} \to C'_{ij}$ be two representations of GL_2 in $C(1,3)$. Then Hopf algebra actions*

$$c_{ij} \cdot v = \sum_k C_{ik} v C^*_{kj} \tag{46}$$

and

$$c_{ij} * v = \sum_k C'_{ik} v C'^*_{kj} \tag{47}$$

are equivalent if and only iff

$$C'_{11} = uC_{11}u^{-1}\alpha_1,\ C'_{12} = uC_{12}u^{-1}\alpha_2,$$

$$C'_{21} = uC_{21}u^{-1}\alpha_1,\ C'_{22} = uC_{22}u^{-1}\alpha_2, \tag{48}$$

for some nonzero complex numbers α_1, α_2 and invertible $u \in C(1,3)$. If $u = 1$, then the actions coincide.

Proof. The proof follows like in Theorem 2, reference [1].

In terms of modules, this result says that the equivalence of representations means that corresponding modules V_1, V_2 are related by formula $V_1 \simeq V_2 \otimes U$, where U is any one dimensional module.

5 Invariants and the operator algebra.

For a given representation $c_{ij} \to C_{ij}$ we denote by $\Re$ an *operator algebra* i.e. a subalgebra of $C(1,3)$ generated by C_{ij}. Recall that the algebra of invariants of an action is defined in the following way

$$Inv = \{v \in C | \forall h \in H \quad h \cdot v = \varepsilon(h)v\}. \tag{49}$$

being H any Hopf algebra and $\epsilon(h)$ the corresponding counit. On the other hand the Invariant algebra equals the centralizer of $\Re$ in $C(1,3)$.

In this Section we present five ingredients for every representation of the quantum GL_2 by Dipper-Donkin with *nonzero perturbation*: the values of C_{ij}, the matrix form of the operator algebra $\Re$, its dimension, the invariants of the inner action defined by this representation I, and the value of the quantum determinant.

To obtain the full classification presented in reference [10], from where we extract the representations given in this Section, Theorem 1 and Figure 2 are used. Additional information (i.e. $\Re$, I, etc) are derived from Theorem 2 and Corollary 1. Theorem 3 is intended to address the question about minimal

nonequivalent representation for GL_2 by Dipper and Donkin on $C(1,3)$, this question remains open, so far.

CASE 1).

$$\begin{array}{l} d = diag(q^2, q, q, 1) \\ C_{12} = qe_{13} - \mu e_{24} \\ C_{21} = -\mu e_{21} + e_{43} \\ C_{11} = diag(1, q^{-1}, 1, q^{-1}) \\ C_{22} = diag(q^2, q^2, q, q) - q\mu e_{23} \end{array} \quad ; \quad \Re \cong \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \otimes \begin{pmatrix} * & * \\ 0 & * \end{pmatrix};$$

$$dim\Re = 9 \;\; ; \quad \text{Invariants} \cong C.$$

This corresponds to CASE 4) in Theorem 1.

CASE 2).

$$\begin{array}{l} d = diag(q^2, q, q, 1) \\ C_{12} = qe_{12} + \mu e_{34} \\ C_{21} = \mu e_{31} + e_{42} \\ C_{11} = diag(1, 1, q^{-1}, q^{-1}) \\ C_{22} = diag(q^2, q, q^2, q) + q\mu e_{32} \end{array} \quad ; \quad \Re \cong \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \otimes \begin{pmatrix} * & * \\ 0 & * \end{pmatrix};$$

$$dim\Re = 9 \;\; ; \quad \text{Invariants} \cong C.$$

This corresponds to CASE 5) in Theorem 1.

For CASE 6), in Theorem 1, we find that there exist no set $\{c_{ij}, d\}$, $1 \leq i, j \leq 2$, that fulfills the algebra in Figure 1.

All the possible representations for the quantum GL_2 by Dipper- Donkin are reported elsewhere [10]. From there, we can deduce the following.

a) Only for nonzero perturbation representations of GL_2, $dim\Re = 9$; which turns out to be the maximal possible dimension of $\Re$ for the action of

GL_2 on $C(1,3)$.

b) The maximal dimension for I is 6.

c) There is only one, zero perturbation, possible case in the set of all representations of GL_2 by Dipper-Donkin, on $C(1,3)$, for which $dim\Re$=8. This is as follows

$$\begin{array}{ll} d = diag(q^2, q, 1, 1) & C_{12} = \alpha e_{12} + \beta e_{23} + \gamma e_{24} \\ C_{21} = 0 & C_{11} = \mathbf{1} \\ C_{22} = q^2 e_{11} + q e_{22} + e_{33} + e_{44}. & \end{array}$$

Besides, for this case we know that

$$\Re = \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & \epsilon & 0 \\ 0 & 0 & 0 & \epsilon \end{pmatrix} \;;\;\; I = C.$$

d) There is only one, zero perturbation, possible case in the set of all representations of GL_2 by Dipper-Donkin, on $C(1,3)$, for which $dim\Re$=3. This is as follows

$$\begin{array}{ll} d = diag(q^2, q, q, 1) & C_{12} = 0 \\ C_{21} = 0 & C_{11} = e_{11} + \alpha_1 e_{22} + m\alpha_2 e_{33} + \alpha_3 e_{44} \\ C_{22} = q^2 e_{11} + q\alpha_1^{-1} e_{22} + \alpha_2^{-1} e_{33} + \alpha_3^{-1} e_{44}. & \end{array}$$

Besides, for this case we know that

$$\Re = \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & \epsilon & 0 & 0 \\ 0 & 0 & \epsilon & 0 \\ 0 & 0 & 0 & * \end{pmatrix} \;;\;\; I = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & \gamma & 0 \\ 0 & \delta & \epsilon & 0 \\ 0 & 0 & 0 & * \end{pmatrix}.$$

e) For the representations wherein d=$diag(\alpha, q^2, q, 1)$, $\alpha \neq 0$, q^{-1}, 1, q, q^2, q^3, which correspond to CASE 5) in Theorem (1) [1], always $dim\Re$=6 and $I = C \oplus C$.

f) For the representations wherein $d = diag(q^3, q^2, q, 1)$ which correspond to CASE 4) in Theorem 1 [1], always $dim\Re$=7 and $I = C$.

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References

[1] Vladislav Kharchenko, Jaime Keller and Suemi Rodríguez-Romo: "Actions of $GL_q(2, C)$ on $C(1, 3)$ and its four dimensional representations" to appear in Communications in Algebra.

[2] Dipper, R., Donkin, S.: Quantum GL_n. Proc. London Math. Soc.(3), **63,** 165–211(1991).

[3] Manin, Yu.I.: Quantum groups and non-commutative geometry. CRM, Université de Montreal 1988.

[4] Cohen, M., and Fishman, D.: Hopf algebra actions. Journal of Algebra, **100**, 363-379 (1986).

[5] Shnider, S., and Stenberg, S.: Quantum Groups. International Press, Boston, 1993.

[6] Koppinen, M. : A Skolem-Noether theorem for coalgebra measurings, Arch. Math. 57 (1991), 34-40.

[7] Montgomery, S. : Hopf Algebras and their Actions on Rings, CBMS, AMS Regional Conference Series in Mathematics, w. 82, 1992.

[8] Min, K.: Simple modules over the coordinate rings of quantum affine space. Bull. Austral. Math. Soc. **52**, 231-234 (1995).

[9] Smith, S.P.. Quantum groups: An introduction and survey for ring theorists. In: Montgomery, S.(ed.). Noncommutative rings. NSR Publ. **24**, 131-178. New York: Springer 1992.

[10] Rodríguez-Romo, S.: math.QA/9805017